Application of Sieve Methods to Prime Numbers

Jan Kristian Haugland
Brasenose College
Trinity 1998

Thesis submitted for the degree of Doctor of Philosophy in the University of Oxford

(Corrected version, April 2000)

ABSTRACT

In this thesis, we improve on the best known results so far related to two famous conjectures on the distribution of prime numbers.

The twin prime conjecture asserts that there are infinitely many pairs of prime numbers with difference 2, and probabilistic arguments suggest that the number of such pairs below x is indeed asymptotically $2C_2 \frac{x}{\log^2 x}$ where C_2 = 0.66016... is known as the ”twin prime constant”. With the use of sieve methods, it is possible to give an upper bound of the form $(1+o(1))kC_2 \frac{x}{\log^2 x}$ for the number of such pairs below x. To find a proof with a smallest possible k is interesting in its own right, and also because it indicates how far our present methods are from being good enough that we can give a non-trivial lower bound. Here, we combine known methods, like the conditional improvement of the Bombieri-Vinogradov Theorem in the linear sieve and the switching principle, with a method that Heath-Brown has previously used in the treatment of the second problem in this thesis. The idea is to estimate the characteristic function for primes from above or below (depending on the problem) by a function which is easier to handle with the available sieve methods. The combination of three rather different methods leads to quite extensive calculations, and also to the necessity of deriving two new results (well, at least one non-trivial result) concerning finite subsets of [0, 1] with sum 1. While the previously best result was k = 6.8354 by Wu, we find that we can do k = 6.8325.

The second conjecture that we consider says that there is always a prime between two consecutive positive square numbers. There exist sieve methods that allow us to prove that there is always an integer between x and $x + x^{\frac{1}{2}+o(1)}$ with a prime factor bigger than $x^{1-\theta}$ for rather small θ. For example, Harman has given $\theta = \frac{1}{20}$ in an unpublished manuscript. We follow much the same path as other authors, but do a detailed analysis of the 4-dimensional terms of the iterated Buchstab identity (applied to the relevant sequence). It appears that $\theta = \frac{1}{25}$ is permissible.

ACKNOWLEDGEMENTS

I am indebted to Dr. Roger Heath-Brown for excellent supervision throughout my three years in Oxford. Also thanks to Prof. Bryan Birch for being a ”substitute supervisor” and for discussion of a couple of mathematical problems.

Thanks to the Norwegian Research Council for financial support during most of my time here.

To the British and international Othello community: Thanks for many many great games!

[Added September, 2009] Thanks to Dr. Yannick Saouter for creating the PDF-file from my old Scientific Word-file, thus making the thesis far more accessible.

CONTENTS